

Instability of a conducting fluid cylinder in the presence of an axial current, a longitudinal magnetic field and a coaxial conducting cylinder

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With 7 figures in the text

ABSTRACT

The stability of an infinitely long, inviscid, incompressible fluid cylinder of finite electrical conductivity in the presence of an axial current, a longitudinal magnetic field and an infinitely long coaxial conducting sheath placed at some distance from the fluid cylinder is investigated. The axial current is assumed to be maintained constant by external sources. The effect of surface tension is included. In the case when the fluid is mercury and the axial current and the longitudinal magnetic field are of the order of a few hundred amperes and a few hundred gauss respectively, it is shown that there exists a critical product of axial current and longitudinal magnetic field which determines whether the $m=0$ mode or the $m=1$ mode will be manifested first in the instabilities.

The results of this theory are compared with those obtained by previous authors for the same geometry but with the assumption of infinite electrical conductivity for the fluid.

I. Introduction

The stability of an infinitely long, inviscid and incompressible fluid cylinder under the action of surface tension has been studied by Rayleigh [1]. The hydro-magnetic analogue of this problem (excluding surface forces) was considered, among other problems, by Chandrasekhar and Fermi [2] who showed that the presence of an axial magnetic field frozen inside the medium increases the stability of the cylinder for an axi-symmetric perturbation. Simon [3] generalised the results of Chandrasekhar and Fermi for three dimensional perturbation when the magnetic field is also present outside the infinitely conducting fluid cylinder. The extension of the stability considerations to the situation when the fluid is compressible and carries an axial current is of particular interest because of its application to the study of the behaviour of constricted gas discharges which are pinched under the mutual interaction of discharge current filaments. Kruskal

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and Schwarzschild [4] and Taylor [5] showed that a highly constricted gas discharge is unstable. Further Taylor [6], Shafranov [7], Rosenbluth [8] and Kruskal and Tuck [9] showed that the stability of a pinched discharge is increased (1) if it contains an axial magnetic field frozen in the medium and (2) if it is surrounded by a coaxial conducting sheath of infinite electrical conductivity.

In the study of hydromagnetic stability of a given configuration, it is usually assumed that the fluid in question has infinitely large electrical conductivity. This assumption is valid (1) in a highly ionised plasma and (2) under astrophysical conditions. In the case of experiments carried out in the laboratory with mercury as the conducting fluid the hydromagnetic coupling is too weak to permit the assumption of infinite electrical conductivity. However, if the electric current is maintained by external sources at a constant value an alternative simplification is allowed. One can neglect quantities proportional to the electrical conductivity in the induction equation and supplement the problem by an additional constraint on the system by demanding, for example, that the total current is constant when the system is perturbed. Without going into the question of uniqueness of the solutions obtained by such a procedure that changes the structure of the differential equations, it was shown recently by Murty [10] that the method enables one to explain satisfactorily the axis-symmetric instabilities observed by Dattner *et al.* [11] and Dattner [12] in their experiment which was a liquid conductor model of pinched discharges.

In the above-mentioned experiment mercury was allowed to fall freely under gravity from a circular hole of 0.2 cm radius in a tank, in the form of a jet of 14 cm length. When an axial current passes through the jet, sausage instabilities ($m=0$ mode) are observed, and when a longitudinal magnetic field was also present helical instabilities ($m=1$ mode) were observed.

It is the aim of the present paper to generalise the theory developed in [10] for a non-axisymmetric perturbation in the presence of a longitudinal magnetic field and also an external coaxial conducting cylinder and compare the results with the above-mentioned experiment of Dattner *et al.* A comparison will also be made between the results of the present paper obtained with the assumption of vanishing electrical conductivity for the fluid with those obtained for infinitely conducting fluids by previous authors.

II. Formulation of the problem

(a) Basic assumptions and equations

Let us consider a static, infinitely long, inviscid and incompressible fluid cylinder of radius R_0 immersed in a uniform magnetic field H_0 , extending throughout space and directed parallel to the axis of the fluid cylinder. A current of constant volume density is also passing along the cylinder. The directions of current and the magnetic field from external sources are chosen to be parallel so that the resultant magnetic lines of force form right helices everywhere. Let the fluid be non-magnetic, have vanishingly small electrical conductivity and be surrounded by an infinitely conducting coaxial cylinder placed at a radial distance equal to ΛR_0 from the axis. We shall further assume that the region between the fluid cylinder and coaxial conducting cylinder is a vacuum. The

effect of external forces due to gravity will be ignored¹ but the effect of surface tension will be included. In C.G.S. electromagnetic units, neglecting the displacement current, the basic equations are,

$$e \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{H}, \quad (2.1)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.2)$$

$$\nabla \times \mathbf{H} = 4\pi \mathbf{j}, \quad (2.3)$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}, \quad (2.4)$$

$$\mathbf{j} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{H}), \quad (2.5)$$

$$\nabla \cdot \mathbf{H} = 0, \quad (2.6)$$

$$\text{and} \quad I = \text{constant}. \quad (2.7)$$

The symbols in eqns. (2.1) to (2.6) have their usual meaning [13]. Eqn. (2.7) is a constraint on the system in the present case which indicates that the axial current in the fluid cylinder is maintained constant by external sources.

(b) *Equilibrium*

The equilibrium state of the fluid cylinder is described by

$$0 = -\nabla \hat{p}_e + \mathbf{j}_e \times \mathbf{H}_e,$$

and eqns. (2.3) and (2.7). If we choose a system of right circular cylindrical coordinates with the z -axis coinciding with the axis of the fluid cylinder and if the current density inside the fluid is given by

$$\mathbf{j}_e = (0, 0, j_0), \quad (2.8)$$

the fluid cylinder will be in equilibrium with the pressure distribution given by

¹ The inclusion of the gravity can have a nontrivial effect on the solution of this stability problem only when fluid velocity is a function of the position along the axis of the fluid cylinder. In the case of Dattner's experiment there is in fact a variation of the fluid velocity along the axis of the fluid cylinder and the equilibrium distribution of the physical quantities like the current density, the pressure and the magnetic field are functions of not only the r -coordinate but also of the z -coordinate. Since the study of stability of such an equilibrium presents enormous difficulties, we shall study only an idealised model by assuming that the fluid cylinder is in static equilibrium defined by functions depending only on the r -coordinate. We shall see that even this simplified model of Dattner's experiment is not so bad as is might appear at first sight.

$$p_e = \pi j_0^2 (R_0^2 - r^2) + T/R_0, \quad \text{when } r \leq R_0, \quad (2.9)$$

where T = surface tension of the fluid. The pressure outside the fluid is zero because of the basic assumptions we have made.

(c) *Perturbation and criterion of instability*

In order to determine whether the equilibrium state given by eqns. (2.8) and (2.9) is stable or not, we shall assume that the fluid cylinder is subjected to a perturbation so that its deformed surface is given in circular cylindrical co-ordinates by

$$r = R + a \cos (-m\theta + kz + \varepsilon), \quad (2.10)$$

where ε is a phase factor and a may be a function subject to the condition $|a|/R \ll 1$, so that we can neglect quantities $\sim (a/R)^2$ in the linear approximation. The basis for the assumption regarding the form of the perturbed surface as given by eqn. (2.10) is that a general form of the perturbed boundary of the fluid cylinder can be obtained by a combination of such terms as long as nonlinear effects are ignored. It is therefore sufficient to discuss the effect of one typical term as the one given in eqn. (2.10). It is no loss of generality in the present discussion of stability of the equilibrium to assume that the phase factor ε in eqn. (2.10) is equal to zero. We further assume that a is periodic in time so that

$$r = R + a \cos \varphi, \quad (2.11)$$

$$\text{and} \quad a = a_0 \cos \omega t, \quad (2.12)$$

$$\text{where} \quad \varphi \equiv -m\theta + kz. \quad (2.11 a)$$

The purpose of the stability theory is to calculate the values of ω . If ω is found to have an imaginary value for any value of m and k , we conclude that the fluid cylinder is unstable for arbitrarily small perturbations of the boundary.

Since the fluid is assumed to be incompressible one can show that R and R_0 are related by

$$R_0^2 = R^2 + a^2/2. \quad (2.13)$$

Eqn. (2.13) shows that in the linear approximation one can ignore the difference between R and R_0 .

III. Solution for ω

(a) *Current density distribution*

If we combine eqns. (2.4) and (2.5) we obtain

$$\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \left[\frac{\mathbf{j}}{\sigma} - \mathbf{v} \times \mathbf{H} \right], \quad (3.1)$$

which is the equation for the magnetic induction. If the fluid is an ideal conductor, eqn. (3.1) leads to the well-known equation for induction for frozen magnetic lines of force. On the other hand, if the fluid is a poor electrical conductor we can assume that

$$\nabla \times \mathbf{j} = 0, \quad (3.2)$$

since all other quantities in eqn. (3.1) are finite. The current density vector is also solenoidal (see eqn. (2.3)) and therefore eqn. (3.2) shows that we can define a scalar function ψ such that

$$\mathbf{j} = \hat{\mathbf{z}} j^* + a \nabla \psi, \quad (3.3)$$

where ψ satisfies
$$\nabla^2 \psi = 0, \quad (3.4)$$

and j^* is the mean current density flowing in the z -direction. The solutions of eqn. (3.4) (in circular cylindrical coordinates) that are regular in the domain occupied by the fluid can be written as

$$\psi = C I_m(kr) \sin \varphi, \quad (3.5)$$

where $I_m(kr)$ is the modified Bessel function and C is an arbitrary constant. We note here that the eqn. (3.3) is formally the same as the one obtained by Chandrasekhar and Fermi [2] for the distribution of the magnetic field in connection with the study of the stability of a fluid cylinder in the presence of an axial magnetic field frozen in the medium, and the expression for ψ in the eqn. (3.5) is a generalisation of their eqn. (75) in [2]. Since we are interested only in the linear approximation, we have neglected higher order terms in eqn. (3.5).

The boundary condition for the solution of eqn. (3.4) is that the normal component of the current density vector at the perturbed boundary is zero, or¹

$$(\mathbf{j} \cdot \mathbf{n})_b = 0, \quad (3.6)$$

where $\mathbf{n}$ is the unit vector normal to the perturbed boundary and the suffix b indicates that the quantity concerned shall be evaluated at the perturbed boundary i.e. at $r = R + a \cos \varphi$. One can show that $\mathbf{n}$ is given by

$$\mathbf{n} = (1, 0, 0) + (0, -am/R, ak) \sin \varphi. \quad (3.7)$$

Using eqns. (3.3), (3.5), (3.6) and (3.7) we find that

$$C = -j^*/I'_m(kR), \quad (3.8)$$

where (and in the following) the prime to the modified Bessel function indicates differentiation with respect to the argument. Due to the constraint that the total current flowing along the fluid cylinder is constant (eqn. (2.7)), we find that

$$j^* = j_0. \quad (3.9)$$

¹ The eqn. (4.6) of [10] should read as $(\mathbf{j} \cdot \mathbf{n})_{r=R+a \cos \varphi} = 0$.

If we combine eqns. (3.3), (3.5), (3.8) and (3.9) we can write the perturbed current density distribution as

$$\mathbf{j} = (j_r, j_\theta, j_z), \quad (3.10)$$

$$\left. \begin{aligned} \text{where} \quad j_r &= -a k j_0 \frac{I_m(kr)}{I'_m(kR)} \sin \varphi, \\ j_\theta &= \frac{a m j_0}{r} \frac{I_m(kr)}{I'_m(kR)} \cos \varphi, \\ \text{and} \quad j_z &= j_0 - a k j_0 \frac{I_m(kr)}{I'_m(kR)} \cos \varphi. \end{aligned} \right\} \quad (3.11)$$

We note that if we put $m=0$ in eqns. (3.11) we obtain eqns. (4.8) of [10].

(b) Magnetic field distribution

In order to obtain the perturbed magnetic field distribution, $\mathbf{H}_i$, inside the fluid cylinder from eqns. (2.3) and (2.6) we assume that we can separate the variables r, θ and z , and write

$$\mathbf{H}_i = (H_{ri}, H_{\theta i}, H_{zi}), \quad (3.12)$$

$$\left. \begin{aligned} \text{where} \quad H_{ri} &= H_{ri}(r) \sin \varphi, \\ H_{\theta i} &= H_{\theta i}(r) \cos \varphi + 2\pi j_0 r, \\ \text{and} \quad H_{zi} &= H_{zi}(r) \cos \varphi. \end{aligned} \right\} \quad (3.13)$$

The phases in eqns. (3.13) for the magnetic field are so chosen as to be consistent with the phases of the current density distribution given by eqns. (3.11). The reason for the particular choice of the θ -component of the magnetic field in eqns. (3.13) is that the z -component of the current density has a constant mean value which gives, inside the fluid cylinder, a toroidal component of the magnetic field increasing proportionally with the distance from the axis. If we eliminate the r - and θ -components of the magnetic field between the eqns. (2.3) and (2.6) we find, using eqns. (3.11), that $H_{zi}(r)$ satisfies the equation

$$\left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \right) - \left(k^2 + \frac{m^2}{r^2} \right) \right] H_{zi}(r) = 0, \quad (3.14)$$

whose well-behaved solution in the domain of interest can be written as

$$H_{zi}(r) = A I_m(kr), \quad (3.15)$$

where A is an arbitrary constant. Using eqns. (2.3), (3.11), (3.13) and (3.15), one can show that the radial parts of the remaining components of the magnetic field inside the fluid cylinder are given by

$$\left. \begin{aligned} H_{r1}(r) &= A I'_m(kr) + \frac{4\pi a m j_0}{I'_m(kR)} \frac{I_m(kr)}{kr}, \\ H_{\theta 1}(r) &= -\frac{m A I_m(kr)}{kr} - \frac{4\pi a j_0}{I'_m(kR)} I'_m(kr). \end{aligned} \right\} \quad (3.16)$$

and In the region outside the fluid cylinder and inside the coaxial conducting cylinder the magnetic field vector is both irrotational and solenoidal and hence can be obtained as the gradient of a harmonic function. One can show that the general form of the magnetic field distribution, $\mathbf{H}_0$, outside the fluid cylinder is given by

$$\mathbf{H}_0 = [H_{r0}, H_{\theta 0}, H_{z0}], \quad (3.17)$$

$$\left. \begin{aligned} \text{where} \quad H_{r0} &= k [B_1 I'_m(kr) + B_2 K'_m(kr)] \sin \varphi, \\ H_{\theta 0} &= -\frac{m}{r} [B_1 I_m(kr) + B_2 K_m(kr)] \cos \varphi + \frac{2\pi j_0 R^2}{r}, \\ \text{and} \quad H_{z0} &= k [B_1 I_m(kr) + B_2 K_m(kr)] \cos \varphi, \end{aligned} \right\} \quad (3.18)$$

where $K_m(x)$ is that modified Bessel function that is regular at $x = \infty$ and B_1 and B_2 are arbitrary constants. The magnetic field should be represented by real functions and therefore we assume that $k \geq 0$ in eqns. (3.18). Since the external cylinder at a radial distance equal to ΛR_0 is assumed to be an ideal conductor, the radial component, H_{r0} , of the magnetic field must vanish at $r = \Lambda R_0$, which shows that

$$\frac{B_1}{B_2} = -\frac{K'_m(\Lambda k R_0)}{I'_m(\Lambda k R_0)}. \quad (3.19)$$

We assumed that the fluid cylinder has only a volume distribution of current and therefore the magnetic field vector has to be continuous at its boundary. By matching at the boundary the inside and outside solutions of the magnetic field as given by eqns. (3.13) to (3.19), we find that the constant A in eqns. (3.15) and (3.16) is given by

$$A = -\frac{4\pi a m j_0 I_m(kR)}{I'_m(kR)} \left[K_m(kR) - I_m(kR) \frac{K'_m(\Lambda k R)}{I'_m(\Lambda k R)} \right], \quad (3.20)$$

where the difference between R and R_0 has been ignored by virtue of eqn. (2.13). The constants B_1 and B_2 can similarly be determined but their values are not given since they are not required for our purpose. Therefore eqns. (3.13) to (3.20) determine the perturbed magnetic field distribution both inside and outside the fluid cylinder up to terms of order a .

(c) Pressure distribution

If we take the divergence of both sides of eqn. (2.1) and use eqns. (2.2), (2.3), (3.3) and (3.11) we obtain, ignoring higher order terms,

$$\nabla^2 p = -4\pi j_0^2 + 8\pi a k j_0^2 \frac{I_m(kr)}{I'_m(kR)} \cos \varphi, \quad (3.21)$$

which is merely Poisson's equation. Following the method adapted in [10] one can show that the solution of eqn. (3.21) is given by

$$p = \frac{T}{R} + \pi j_0^2 (R^2 - r^2) + 4\pi a j_0^2 \frac{r I'_m(kr)}{I'_m(kR)} \cos \varphi \\ + a \frac{I_m(kr)}{I'_m(kR)} \left[\frac{T}{R^2} (k^2 R^2 + m^2 - 1) - 2\pi j_0^2 R \right] \cos \varphi. \quad (3.22)$$

Eqn. (3.22) represents the pressure distribution at all points in the interior of the perturbed boundary of the fluid cylinder. It has been obtained from the condition that the pressure at a point just inside the boundary is that due to surface tension.

(d) Dispersion relation

In order to obtain the values of ω , we note that the radial part of eqn. (2.1) evaluated at the boundary can be written, using eqns. (2.11) and (2.12), as

$$-\rho \omega^2 a \cos \varphi = \{ -\partial p / \partial r + (\mathbf{j} \times \mathbf{B})_r \}_{b}, \quad (3.23)$$

where the suffix to the braces indicates that the quantity concerned must be evaluated at $r = R + a \cos \varphi$. Using eqns. (3.11), (3.13), (3.15) to (3.20) and (3.22) and including in the electromagnetic force the effect of the homogeneous longitudinal magnetic field produced by external sources, one can show that eqn. (3.23) leads to

$$\omega^2 = \frac{T}{\rho R^3} \frac{x I'_m(x)}{I_m(x)} (x^2 + m^2 - 1) - \frac{4 I^2}{\pi \rho R^4} F_m(x) - \frac{m I H_0}{\pi \rho R^3} \frac{I_m(x)}{I'_m(x)}, \quad (3.24)$$

where $I = \pi j_0 R^2$, $x \equiv kR$ and

$$F_m(x) \equiv 1 + \frac{x}{2} \left\{ \frac{I'_m(x)}{I_m(x)} - \frac{I_m(x)}{I'_m(x)} \right\} - \frac{m^2 I_m(x)}{x I'_m(x)} \left\{ 1 + I_m(x) \left(K_m(x) - \frac{I_m(x) K'_m(\Lambda x)}{I'_m(\Lambda x)} \right) \right\}, \quad (3.25)$$

and H_0 = homogeneous magnetic field parallel to the z -axis produced by external sources. Eqn. (3.24) is the required dispersion relation which gives the angular "frequency" of vibration of the surface of the fluid cylinder for various modes of perturbation given by eqn. (2.11).

IV. Discussion

(a) General results

As was stated in § II (c), the fluid cylinder is unstable for arbitrarily small perturbations for all values of x for which ω^2 assumes a negative value. In

the present solution given by eqn. (3.24) only real values of $x \geq 0$ are allowed and ω^2 is then a real number showing that there are no states of over-stability, a result which is similar to the one obtained by Bernstein *et al.* [14]. If we set $m=0$ in eqn. (3.24), we obtain the eqn. (6.2) of [10]. When $m=0$, the effect of an external magnetic field, H_0 , is zero, which is a consequence of the fact that we have assumed the fluid to be a poor conductor. When $m \neq 0$, we find that only the third term on the right-hand side of eqn. (3.24), containing the contribution due to an external magnetic field, changes sign if m or x changes sign. In the present problem we assumed that the magnetic field produced by external sources is directed parallel to the direction of axial current, and so the fluid cylinder is unstable for that mode $m \neq 0$ which is right-handed in a right circular cylindrical coordinate system. An examination of the current distribution given by eqns. (3.11) shows that the unstable deformation is the one that makes the current density distribution tend to the force-free state at the points that are perturbed away from the axis. One can also verify that such a distribution of current increases the magnetic pressure at a point outside that part of the fluid surface which is perturbed inwards and vice versa and one concludes that the cylinder is unstable for such perturbations. We note here that the first term on the right-hand side of eqn. (3.24) is the one obtained by Rayleigh [1].

(b) Neither external conducting cylinder nor surface tension

If the jet is not enclosed in a conducting cylinder, i.e. if $\Lambda = \infty$ and the fluid has no surface tension, one can show that eqn. (3.24) reduces to

$$\omega_b^2 = -\frac{4I^2}{\pi \rho R^4} \left[1 + \frac{x}{2} \left\{ \frac{I'_m(x)}{I_m(x)} - \frac{I_m(x)}{I'_m(x)} \right\} - \frac{m^2 I_m(x)}{x I'_m(x)} \{ 1 + I_m(x) K_m(x) \} \right] - \frac{m I H_0}{\pi \rho R^3} \frac{I_m(x)}{I'_m(x)}, \quad (4.1)$$

where the suffix b in ω_b^2 indicates the conditions stated in the beginning of this subsection for which the equation is valid. By considering the asymptotic behaviour of the modified Bessel functions one can show

$$\lim_{x \rightarrow 0} L t \omega_b^2 \rightarrow \frac{2I^2}{\pi \rho R^4} (|m| - 1) + O(x), \quad (4.2)$$

$$\text{and} \quad \lim_{x \rightarrow \infty} L t \omega_b^2 \rightarrow -\frac{2I^2}{\pi \rho R^4} \left(1 + \frac{m R H_0}{2I} \right) + O\left(\frac{1}{x}\right). \quad (4.3)$$

The behaviour of ω_b^2 for $x \rightarrow 0$ shows that for infinitely long wavelength disturbances the fluid cylinder is neutral for the $|m|=1$ mode, while it is stable for $|m| > 1$. In this respect the effect of an axial current is similar to that of surface tension [1]. The behaviour of ω_b^2 for $x \rightarrow \infty$ shows that the fluid cylinder is unstable for a disturbance of very small wavelength, but the growth rate of perturbations is finite for all finite values of m , which is in contrast with the infinite growth rate of small wavelength disturbances obtained by Kruskal and Schwarzschild [4] and Taylor [5], who assumed surface current distri-

bution for an infinitely conducting plasma cylinder. Also, Taylor obtained finite growth rate for small wavelength disturbances if the infinitely conducting plasma has a volume current distribution [5]. We see therefore that the growth rate is finite for volume distribution of current and is infinite for surface current distribution. The reason for this behaviour is that the effective mass involved in the motion decreases with wavelength of perturbation, while the surface density of the force remains constant so that the acceleration becomes very large in the limit of vanishingly small wavelengths. It must, however, be noted that the fluid equations and the perturbation technique are no longer valid at the limit of vanishing wavelength of the perturbation.

As mentioned earlier, the asymptotic values of ω_b^2 , given by eqns. (4.2) and (4.3), show that all the modes are unstable for small wavelengths, while for long wavelengths the modes with $|m| > 1$ are stable. Therefore a given mode $|m| > 1$ passes through a neutral state as the wavelength of the perturbation decreases from infinitely large values to small values. The variation of ω_b^2 for $m=2$ when $H_0=0$ is shown in Fig. 1, where it is seen that ω_b^2 has a zero when $x \approx 4$. The neutral point for a general value of $m > 1$ can be obtained by considering the asymptotic behaviour of the modified Bessel functions in the range $1 \ll x \ll \infty$. One can show that in the above limits and when $H_0=0$, $\omega_b^2=0$ when $m^2 \approx x/2$.

The reason for the existence of a neutral state for some wavelength when $m=2$ is qualitatively as follows. An examination of the current density distribution given by eqns. (3.11) shows that for deformations of finite wavelength and $m=2$ there are both parallel components (z -components) and anti-parallel components (θ -components) of the current. The former components attract each other, while the latter repel, so that there can be a combination of these current configurations with the net force equal to zero. The situation for $m=2$ is shown schematically in Fig. 2.

It is interesting to compare the above results with those obtained by Taylor [5], who showed that a compressible, infinitely conducting fluid cylinder with surface current distribution is neutral to a perturbation of a wavelength corresponding to $x \approx m^2$ when $m \gg 1$. He has also shown that such a neutral state does not exist when the fluid has a volume current distribution. In fact the fluid cylinder is completely stable for all the modes with $m > 1$ when the current is evenly distributed in the whole cross-section of the cylinder and this result is independent of the compressibility of the medium. We see therefore that the behaviour of the modes with $m > 1$ is strongly dependent on the conductivity of the fluid. Although the volume distribution of the current gives greater stability to the (infinitely conducting) fluid cylinder than a surface distribution of current (in the absence of frozen axial magnetic field), the modes with $m > 1$ are unstable if the conductivity of the fluid is small and the currents are maintained by external sources while they are completely stable if the conductivity is infinite as considered by Taylor.

An examination of the limiting form of ω_b^2 when $x \rightarrow 0$ (eqn. 4.2)) shows that when $m=2$ the frequency of vibration, $\nu = \omega_b/2\pi$, of the surface is given by

$$\nu = \frac{\sqrt{2}}{2\pi R} \frac{H_\theta(R)}{\sqrt{(4\pi\rho)}}, \quad (4.4)$$

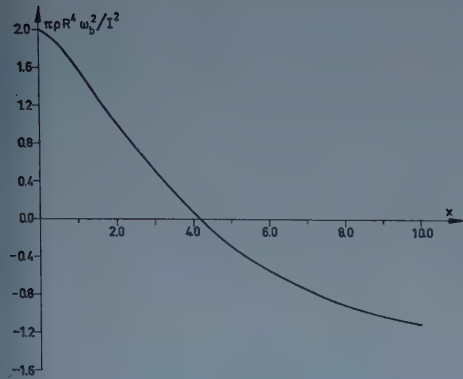


Fig. 1.

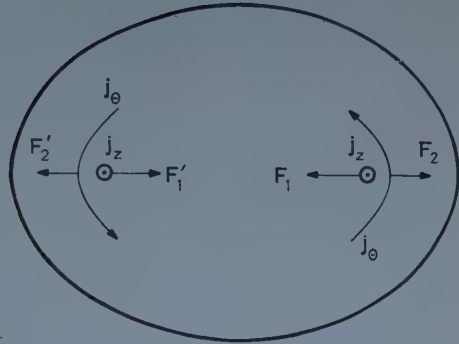


Fig. 2.

Fig. 1. Variation of $\pi \rho R^4 \omega_c^2 / I^2$ for the mode $m=2$ with $x = kR$. The points above the abscissa indicate oscillation of the fluid cylinder for the corresponding values of x with the mode $m=2$. The point where the curve cuts the abscissa corresponds to a critical wavelength for which the fluid cylinder is neutral to a perturbation of that wavelength of the mode $m=2$.

Fig. 2. Effect of a perturbation of finite wavelength with $m=2$ without any external magnetic field. The figure shows qualitatively the cross-section of the perturbed fluid cylinder. The current distribution has parallel components (z -components), whose mutual attraction is denoted by F_1 and F'_1 ; the anti-parallel components (θ -components) repel each other, whose effect is denoted by F_2 and F'_2 . These forces can balance each other for a certain wavelength of the perturbation so that the fluid cylinder is neutral for that wavelength of the perturbation.

where $H_\theta(R)$ is the toroidal component of the magnetic field at the boundary of the cylinder due to the axial current. Eqn. (4.4) suggests that the frequency, ν , is roughly equal to the inverse of the time of travel of an Alfvén wave around the circumference of the fluid cylinder. It is also interesting to note from eqn. (4.3) that, when $x \rightarrow \infty$, i.e. for a disturbance of very small wavelength, the e -folding time of the disturbance is also of the same order as the time of travel of an Alfvén wave around the circumference a result which is similar to the one obtained by Taylor [5] for an infinitely conducting fluid, the only difference being that in his case the e -folding time of the disturbance is $\sqrt{2}$ times the one obtained by us in the above problem. We may conclude from the above results that the time scales defined by Alfvén wave propagation over characteristic distances of the problem play a significant role in both infinitely conducting fluids as well as finitely conducting fluids, provided the currents are maintained by external sources in the latter case.

(c) No external conducting cylinder

If we take into account the effect of surface tension of the fluid but assume that $\Lambda = \infty$, we get from eqn. (3.24), writing $\omega^2 \equiv \omega_c^2$

$$\omega_c^2 = \frac{T}{\rho R^3} \frac{x I'_m(x)}{I_m(x)} (x^2 + m^2 - 1) + \omega_b^2, \quad (4.5)$$

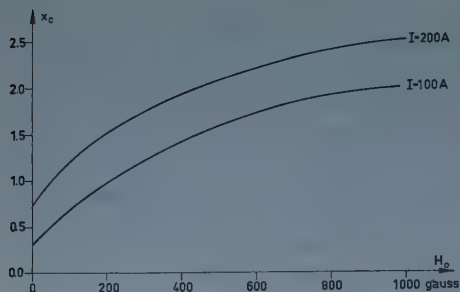


Fig. 3.

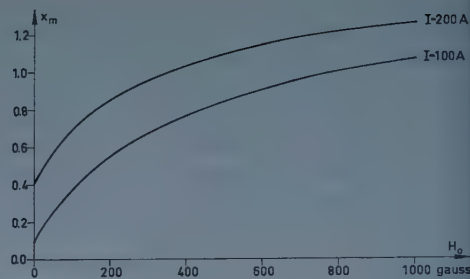


Fig. 4.

Fig. 3. Variation of x_c with H_0 , the homogeneous magnetic field produced by external sources, for two values of the axial current flowing in a mercury cylinder of 0.2 cm radius. The quantity x_c is proportional to the wave number for which the fluid cylinder is neutral to the perturbation of the mode $m = 1$. The neutral state arises because of the stabilising action of surface tension of the fluid.

Fig. 4. Variation of x_m with H_0 , the homogeneous magnetic field produced by external sources for two values of axial current flowing in a mercury cylinder of 0.2 cm radius. The quantity x_m is proportional to the wave number of the perturbation of the $m = 1$ mode for which the fluid cylinder has a maximum growth rate. The existence of the wavenumber with a maximum growth rate for perturbation of the mode $m = 1$ is due to the presence of surface tension of the fluid (mercury in the present case).

where ω_b^2 is given by eqn. (4.1). The first term in eqn. (4.5) is due to the effect of surface tension which has a stabilising effect for all wavelengths when $m \geq 1$ and can give rise to instability for some range of wavelengths (less than the circumference of the fluid cylinder) when $m = 0$, a result which was discussed by Rayleigh [1]. When $x \rightarrow \infty$, that is, when the wavelength tends to zero, the term in eqn. (4.5) containing the effect of surface tension tends to infinity for all values of m . The stabilising effect of surface tension at small wavelengths is similar to that of an axial magnetic field frozen into an infinitely conducting fluid cylinder [6, 7, 8, and 9]. When $m = 1$ and $x \rightarrow 0$, one can show that $\omega_c^2 \rightarrow 0$ and is negative for small values of x . Since $\omega_c^2 > 0$ for $x \rightarrow \infty$, due to the effect of surface tension, we see that ω_c^2 has a zero for some value of x which we may call x_c and has a minimum at some value of x which we may call x_m . (The suffix m in x_m should not be confused with the same symbol denoting the mode of perturbation.) The values of x_c and x_m for various values of axial current and longitudinal magnetic field for the mercury cylinder of 0.2 cm radius for the mode $m = 1$ are obtained by a graphical method and are shown in Figs. 3 and 4. We see from these figures that both x_c and x_m increase with the current and the magnetic field which means the limit of stability moves towards smaller wavelengths with increasing values of current and magnetic field. The ratio x_c/x_m is approximately constant equal to 1.9 in the range of the values of the current and the magnetic field shown in Figs. 3 and 4.

Due to the stabilising effect of surface tension all modes with $m > 1$ are stable for the values of the currents and the magnetic fields used in the experiment of Dattner [12] and the only modes that are not stabilised are those with $m = 0$ and 1. The behaviour of the mode $m = 0$ has been discussed in [10]. For $m = 1$

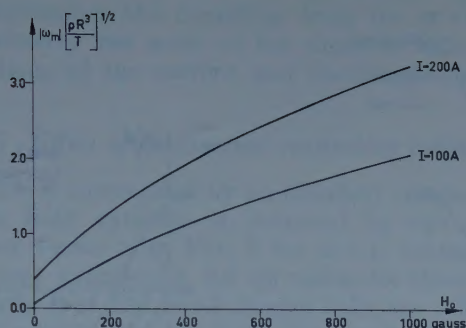


Fig. 5. Variation of the maximum growth rate, $|\omega_m|$, (in non-dimensional units) for the $m=1$ mode of perturbation of the mercury cylinder of 0.2 cm radius with the homogeneous magnetic field produced by external sources for two values of the axial current. ρ is the density of the fluid (mercury); T is the surface tension; R is the radius of the cylinder.

the non-dimensional maximum growth rate,¹ $(\rho R^3/T)^{1/2} |\omega_m|$, corresponding to x_m of Fig. 4 for given values of axial current and longitudinal magnetic field for the mercury cylinder of 0.2 cm radius (and $T=487$ dyne cm^{-1} and $\rho=13.6$ g cm^{-3}) are shown in Fig. 5. The experimental value of the same nondimensional growth rate for the $m=1$ mode which is seen to develop in the experiment of Dattner [11, 12] with a longitudinal magnetic field of 300 gauss and an axial current of 100 A is 1.8 ± 0.6 whose mean value is about twice larger than the theoretical value shown in Fig. 5. The experimental value is obtained from the measurements on deformations which can by no means be considered as small perturbations of the fluid cylinder. Taking into consideration this fact and the possible errors in actual value of the surface tension which can be smaller than that employed in the theoretical calculation the agreement between theory and experiment within a factor of 2 is not unreasonable. The observed large value of the growth rate may be due to non-linear effects but since there are other sources of error due to gravity and viscosity which are not included in the theory it is difficult at this stage to give a reasonable explanation for the discrepancy between theoretical and experimental value.

(d) Relative growth rates of the $m=0$ and the $m=1$ modes

We have seen before that the $m=0$ mode is unaffected by the presence of a longitudinal magnetic field, while modes $m \geq 1$ can be made unstable. When the effects of the surface tension and the longitudinal magnetic field are included, the growth rate for the $m=1$ mode can be smaller than that for the $m=0$ mode for some range of values of the axial current and the longitudinal magnetic field due to the stabilising action of the surface tension for the $m=1$ mode. If the surface tension were zero, only the modes with $m \geq 1$ will develop for all non-zero values of the longitudinal magnetic field. In Fig. 6 are shown the growth rates for the $m=0$ mode and the $m=1$ mode for various values of

¹ $|\omega_m| \equiv [-\omega_c^2(x_m)]^{1/2}$.

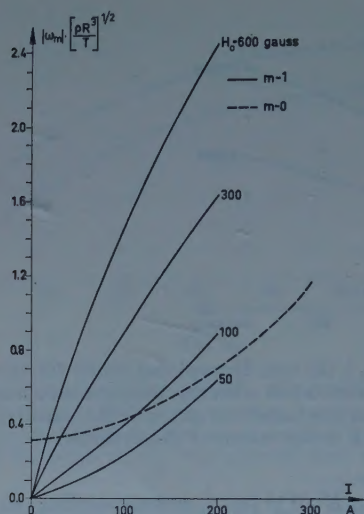


Fig. 6.

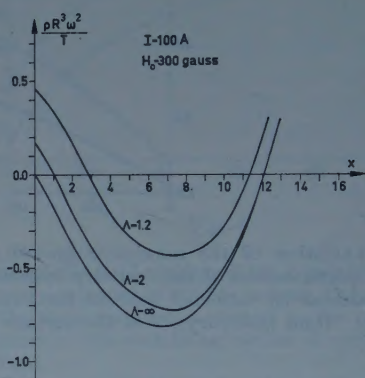


Fig. 7.

Fig. 6. Relative growth rates of the $m=0$ mode and the $m=1$ mode of perturbation of the mercury cylinder of radius 0.2 cm. The ordinate represents the growth rate in non-dimensional units and the abscissa represents the axial current in the cylinder. The continuous curves represent the growth rate of the $m=1$ mode for various values of the homogeneous magnetic field produced by external sources and the discontinuous curve represents the same quantity for the $m=0$ mode of perturbation. (Note that the $m=0$ mode is unaffected by the homogeneous magnetic field.) At the intersection of the continuous curves with the discontinuous curve the values of the growth rates for the $m=1$ mode and the $m=0$ mode are equal. The corresponding values of the axial current and homogeneous magnetic field make a product approximately equal to 10^4 A gauss. In the range of the values of the current and magnetic field shown in the figure the mercury cylinder is stable for all the modes with $m > 1$ due to the stabilising action of the surface tension.

Fig. 7. Effect of an infinitely conducting coaxial cylindrical conductor enclosing the mercury cylinder of 0.2 cm radius. The ordinate represents the square of the "frequency" (in non-dimensional units) of "oscillation" of the fluid cylinder for perturbation of the mode $m=1$ and the abscissa represents the corresponding wavenumber (in non-dimensional units) of the perturbation. The axial current is 100 A and the homogeneous magnetic field is 300 gauss. The effect of surface tension of mercury is included. The values of the parameter Λ are equal to the ratio of the radial distance of the infinitely conducting coaxial cylinder to the radius of the mercury cylinder. The points below the abscissa correspond to instability of the mercury cylinder at the respective wave numbers for the $m=1$ mode of the perturbation. The stabilising effect of the conducting sheath decreases rapidly with increasing values of Λ .

the current and the longitudinal magnetic field. We see from this figure that for a given value of the longitudinal magnetic field, say 100 gauss, the growth rate for the $m=1$ mode is smaller than that for the $m=0$ mode up to the value of the axial current of approximately 100 A. It is seen from the curves in Fig. 6 that the growth rates of the $m=0$ mode and the $m=1$ mode are equal when the product of the axial current and the longitudinal magnetic field is approximately equal to 10^4 A gauss. In the experiment of Dattner [12] the values of the current and the longitudinal magnetic field used give a product which is larger than the above value. It is of some interest to repeat the experiment to verify the "critical product" of the axial current and the longitudinal mag-

netic field which determines the transition from the $m=0$ mode to the $m=1$ mode and also to observe the sense of the unstable helix for the $m=1$ mode relative to the directions of the current and the longitudinal magnetic field.

(e) *Effect of an external conducting cylinder*

If the fluid cylinder is surrounded by an infinitely conducting coaxial cylinder, the stability of the fluid cylinder is governed by eqn. (3.24). The value of $(\rho R^3/T)\omega^2$ is plotted versus x in Fig. 7 for $m=1$, taking $H_0=300$ gauss and $I=100$ A for a mercury cylinder of 0.2 cm radius for three values of Δ , namely, 1.2, 2.0 and ∞ . (Note that the $m \neq 0$ modes only are affected by the presence of an external conductor.) We see from this figure that the external coaxial conductor has a stabilising effect but cannot completely stabilise the fluid cylinder for the above values of the axial current and the longitudinal magnetic field even when placed at a radial distance of 1.2 times the radius of the fluid cylinder. The stabilising effect decreases very rapidly with increasing radial distance of the conductor. The result should be compared with the corresponding situation in an infinitely conducting plasma cylinder which is shown to be stabilised by an external conductor and a frozen axial magnetic field [6, 7, 8 and 9]. The absence of such a marked stabilising effect in our present example is due to the assumption of poor electrical conductivity for the fluid.

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REFERENCES

1. Lord RAYLEIGH, *Theory of Sound*, vol. 2. Dover Publ. (1945).
2. CHANDRASEKHAR, S., and FERMI, E., *Ap. J.* 118, 116 (1953).
3. SIMON, R., *Ap. J.* 128, 375 (1958).
4. KRUSKAL, M. D., and SCHWARZSCHILD, M., *Proc. Roy. Soc. London* 223 A, 348 (1954).
5. TAYLOR, R. J., *Proc. Phys. Soc. London* 70 B, 1049 (1957).
6. TAYLOR, R. J., *Proc. Phys. Soc. London* 70 B, 31 (1957).
7. SHAFRANOV, V. D., *J. Nucl. Energy* 2, 86 (1957).
8. ROSENBLUTH, M. N., *Proc. Third International Conference on Ionisation Phenomena, Venice* (1957).
9. KRUSKAL, M. D., and TUCK, J. L., *Proc. Roy. Soc. London* 245 A, 222 (1958).
10. MURTY, G. S., *Arkiv Fysik* 18, 241 (1960).
11. DATTNER, A., LEHNERT, B., and LUNDQUIST, S., *Proc. Second International Conference on Peaceful Uses of Atomic Energy, Geneva, Vol. 31, P/1708*.
12. DATTNER, A., *Arkiv Fysik* (1961) in press.
13. COWLING, T. G., *Magnetohydrodynamics. Interscience Tracts of Physics and Astronomy*.
14. BERNSTEIN, I. B., *et al.*, *Proc. Roy. Soc.*, 244 A, 17 (1958).

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